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ON THE RELATION BETWEEN PHASE SHIFT ENERGY LEVELS AND THE POTENTIAL

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A method recently developed by GEL'FAND and LEVITAN is adapted to the problem of determining a central potential from the "spectral function" corresponding to a given angular momentum. (The spectral function incorporates the phase shift, binding energies and m additional free parameters if there are m bound states). Two applications are given: An explicit expression is deduced for the totality of potentials with the same phase shift and binding energies as a given potential. Further it is shown that for a given phase shift the position of the bound states is entirely arbitrary and an explicit example is given which illustrates this fact. Implications for the interpretation of scattering data are discussed.

Introduction.

It was shown in a previous paper [1] that a short range central potential is uniquely determined by the phase shift and binding energies for any given angular momentum plus as many additional parameters C_i as there are bound states. For boundary conditions $h\,\varphi(0) + \varphi'(0) = 0$ GEL'FAND and LEVITAN have in a beautiful paper [2] combined phase shift binding energies and the C_i's into one so-called spectral function $\varrho(E)$ and reduced the problem of determining h and the potential to the solution of a linear integral equation. Their method can easily be adopted to the physically interesting case $\varphi(0) = 0$, which will be done in § 1. The resulting integral equation thus provides a method for constructing the totality of potentials corresponding to given phase shift and binding energies.

In § 2 we first exhibit the dependence of these potentials on the parameters C_i. We have solved this problem in a previous paper [3] by elementary means. The present method yields in a simple way the same results in an improved form.

It is a consequence of the theory of GEL'FAND and LEVITAN that the positions of the bound states are independent of the phase shift. This is proved and illustrated by an example in the second part of § 2.

In § 3 we discuss the implications of these results for the interpretation of scattering experiments.

§ 1. Derivation of the Integral Equation.

We consider potentials for which $\int_0^\infty r \,|V(r)|\, dr < \infty$ and $\int_0^\infty r^2 \,|V(r)|\, dr < \infty$. The Schroedinger equation for S-states is

$$\psi''(E, r) + E\,\psi(E, r) = V\psi(E, r). \tag{1.1}$$

We will make use of the following solutions of this equation:

$$\varphi(E, r):\ \varphi(E, 0) = 0;\quad \varphi'(E, 0) = 1 \tag{1.2}$$

and

$$f(k, r):\ \lim_{r\to\infty} e^{ikr} f(k, r) = 1, \tag{1.3}$$

where

$$k^2 = E. \tag{1.4}$$

$\varphi(E, r)$ is for a fixed r an entire function of E, whereas $f(k, r)$ is regular for $Im(k) < 0$ and continuous for $Im(k) \leqq 0$.
For large $|k|$

$$\varphi(k^2, r) \sim \frac{\sin kr}{k}, \tag{1.5}$$

and for large $|k|$ in $Im(k) \leq 0$

$$f(k, r) \sim e^{-ikr}. \tag{1.6}$$

The function $f(k) \equiv f(k, 0)$ determines and is determined by the phase shift and the binding energies, which latter are given by

$$E_l = -\varkappa_l^2;\quad f(-i\varkappa_l) = 0,\quad \varkappa_l > 0\,. \tag{1.7}$$

For real k

$$\varphi(k^2, r) = \frac{1}{2ik}\,[f(k)\,f(-k, r) - f(-k)\,f(k, r)]\,. \tag{1.8}$$

Further we introduce the spectral function $\varrho(E)$ defined by

$$\left.\begin{aligned} \varrho(-\infty) &= 0 \\ \frac{d\varrho}{dE} &= \sum C_l \delta(E - E_l), \quad E < 0 \\ \frac{d\varrho}{dE} &= \frac{1}{\pi}\frac{\sqrt{E}}{|f(\sqrt{E})|^2}, \qquad E \geqslant 0 \end{aligned}\right\} \tag{1.9}$$

where

$$\left.\begin{aligned} C_l &= \left[\int_0^\infty [\varphi(E_l, r)]^2\, dr\right]^{-1} = \frac{2\, i\varkappa_l f'(i\varkappa_l, 0)}{\dot{f}(-i\varkappa_l)}, \\ \dot{f}(-i\varkappa_l) &\equiv \left[\frac{df(k)}{dk}\right]_{k=-i\varkappa_l}. \end{aligned}\right\} \quad (1.10)$$

The spectral function has the property that we have for every square integrable function $F(r)$ the completeness relation

$$\int_0^\infty [F(r)]^2\, dr = \int d\varrho(E) \left[\int_0^\infty dr\, F(r)\, \varphi(E, r)\, dr\right]^2. \quad (1.11)$$

We now consider two potentials $V_1(r)$ and $V(r)$ and the corresponding solutions φ_1, f_1 and φ, f. In analogy with GEL'FAND and LEVITAN we shall establish the equation

$$\varphi(E, r) = \varphi_1(E, r) + \int_0^r dt\, K(r, t)\, \varphi_1(E, t), \quad (1.12)$$

where

$$K(r, t) = -\int d\,[\varrho(E) - \varrho_1(E)]\, \varphi(E, r)\, \varphi_1(E, t). \quad (1.13)$$

For this purpose we first consider the integral

$$\left.\begin{aligned} I &= \int d\varrho(E')\, \varphi(E', r) \int_0^r dt\, \varphi_1(E^1, t)\ \varphi_1(E, t) \\ &= \sum C_l\, \varphi(E_l, r) \int_0^r dt\, \varphi_1(E_l, t)\, \varphi_1(E, t) \\ &+ \frac{1}{\pi} \int_{-\infty}^\infty \frac{k'^2\, dk'}{f(k')\, f(-k')}\, \varphi(k'^2, r) \int_0^r dt\, \varphi_1(k'^2, t)\, \varphi_1(E, t). \end{aligned}\right\} \quad (1.14)$$

We substitute in the first term

$$\varphi(E_l, r) = \frac{f(-i\varkappa_l, r)}{f'(-i\varkappa_l, r)}, \quad (1.15)$$

and in the second term we use Eq. (1.8):

$$\left.\begin{aligned} I &= \sum_l \frac{2\, i\varkappa_l}{\dot{f}(-i\varkappa_l)} f(-i\varkappa_l, r) \int_0^r dt\, \varphi_1(-\varkappa_l^2, t)\, \varphi_1(E, t) \\ &+ \frac{1}{\pi i} \int_{-\infty}^\infty \frac{k'\, dk'}{f(-k')} f(-k', r) \int_0^r dt\, \varphi_1(k'^2, t)\, \varphi_1(E, t). \end{aligned}\right\} \quad (1.16)$$

The second term can be transformed into an integral over a large semicircle in the upper half plane and a sum of residues which cancel the first term. The remaining integral can be evaluated by using the asymptotic expressions (1.5) and (1.6) (compare ref. 1 Eqs. (A 2.26) — (A 2.30)) giving the final result

$$\frac{1}{2}\varphi_1(E,r) = \int d\varrho(E')\,\varphi(E',r)\int_0^r dt\,\varphi_1(E',t)\,\varphi_1(E,t). \quad (1.17)$$

Next, consider the integral

$$J = \int d\varrho_1(E')\,\varphi(E',r)\int_0^r dt\,\varphi_1(E',t)\,\varphi_1(E,t). \quad (1.18)$$

Using the Schroedinger equation we can write

$$\left.\begin{aligned}&\int_0^r dt\,\varphi_1(E',t)\,\varphi_1(E,t)\\ &= \frac{1}{-E'+E}\left[\varphi_1'(E',r)\,\varphi_1(E,r) - \varphi_1(E',r)\,\varphi_1'(E,r)\right].\end{aligned}\right\} \quad (1.19)$$

Substituting $d\varrho_1(E')$ from Eq. (1.9) and eliminating $\varphi_1(E',r)$ by (1.8) and (1.15), respectively, gives, for E not on the positive real axis,

$$\left.\begin{aligned}J = &\sum_l \frac{2i\varkappa_{1l}}{\dot{f}_1(-i\varkappa_{1l})}\,\varphi(-\varkappa_{1l}^2,r)\,\frac{1}{\varkappa_{1l}^2+E}\\ &\times\left[f_1'(-i\varkappa_{1l},r)\,\varphi_1(E,r) - f_1(-i\varkappa_{1l},r)\,\varphi_1'(E,r)\right]\\ &-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{k'\,dk'}{f_1(-k')}\,\varphi(k'^2,r)\,\frac{1}{k'^2-E}\left[f_1'(-k',r)\,\varphi_1(E,r)\right.\\ &\left.-f_1(-k',r)\,\varphi_1'(E,r)\right].\end{aligned}\right\} \quad (1.20)$$

Shifting the path of integration in the second term into the upper half plane gives residues cancelling the first term plus an integral over a large semicircle and a residue at $k' = \sqrt{E}$. The integral over the semicircle can be evaluated by using the asymptotic expression (1.5) and (1.6). The final result is

$$\left.\begin{aligned}&\varphi(E,r) - \frac{1}{2}\varphi_1(E,r)\\ &= \int d\varrho_1(E')\,\varphi(E',r)\int_0^r dt\,\varphi_1(E',t)\,\varphi_1(E,t).\end{aligned}\right\} \quad (1.21)$$

Combining (1.17) and (1.21) and interchanging the order of integration leads to (1.12), (1.13). The same result can be similarly derived when E is on the positive real axis.

From (1.13) one can deduce the following properties of the function $K(r, t)$:

$$\left.\begin{aligned} \frac{\partial^2 K}{\partial r^2} - V(r)\,K &= \frac{\partial^2 K}{\partial t^2} - V_1(t)\,K, \\ K(r, 0) &= 0. \end{aligned}\right\} \tag{1.22}$$

Furthermore substituting (1.12) into the Schroedinger equation and using (1.22) leads to

$$\frac{dK(r, r)}{dr} = \frac{1}{2}\,[V(r) - V_1(r)]. \quad [4]. \tag{1.23}$$

The equations (1.22) and (1.23) characterize $K(r, t)$ uniquely in terms of $V(r)$ and $V_1(r)$.

On the other hand, multiplying (1.12) with $\varphi_1(E, s)$ and integrating with the weight $\varrho(E) - \varrho_1(E)$ gives the Gel'fand-Levitan integral equation

$$K(r, s) + g(r, s) + \int_0^r dt\, K(r, t)\, g(s, t) = 0 \tag{1.24}$$

whose kernel is

$$g(s, t) = \int d\,[\varrho(E) - \varrho_1(E)]\,\varphi_1(E, s)\,\varphi_1(E, t). \tag{1.25}$$

The construction of a potential can now be carried out in the following steps: Phase shift and bound states determine the function $f(k)$ (ref. 1, Eqs. (2.16)—(2.18)). $|f(k)|^2$ and the constants C_l determine by (1.9) the spectral function $\varrho(E)$. Deriving $\varrho_1(E)$ and $\varphi_1(E, r)$ from any convenient comparison potential one can now construct $g(s, t)$ by (1.25). One must then solve the linear integral equation (1.24) for all values of the parameter $r \geqslant 0$ to obtain $K(r, s)$.

Finally the potential is calculated from (1.23).

In order that the integral equation can be solved it is necessary that the corresponding homogeneous equation has no solution. To show this we write down the homogeneous equation, using (1.25):

$$\chi(s) + \int d\,[\varrho(E) - \varrho_1(E)]\,\varphi_1(E, s) \int_0^r dt\, \varphi_1(E, t)\,\chi(t) = 0. \tag{1.26}$$

Multiplying by $\chi(s)$ and integrating over s from 0 to r leads to

$$\left.\begin{aligned} &\int_0^r [\chi(s)]^2\, ds + \int d\varrho(E) \left[\int_0^r ds\, \varphi_1(E, s)\, \chi(s)\right]^2 \\ &- \int d\varrho_1(E) \left[\int_0^r ds\, \varphi_1(E, s)\, \chi(s)\right]^2 = 0. \end{aligned}\right\} \tag{1.27}$$

Applying the completeness relation (1.11) to the function

$$\left.\begin{aligned} F(s) &= \chi(s) \qquad 0 \leqslant s \leqslant r \\ F(s) &= 0 \qquad\quad s > r \end{aligned}\right\} \tag{1.28}$$

reduces (1.27) to

$$\sum_l C_l \left[\int_0^\infty ds\, \varphi_1(E_l, s)\, F(s)\right]^2 + \frac{2}{\pi} \int_0^\infty \frac{k^2 dk}{|f(k)|^2} \left[\int_0^\infty ds\, \varphi_1(k^2, s)\, F(s)\right]^2 = 0. \tag{1.29}$$

Therefore $F(s)$ is orthogonal to all the continuum eigenfunctions of the potential $V_1(r)$ and hence a superposition of the discrete eigenfunctions. Consequently it cannot vanish for $s > r$ in contradiction with (1.28). [5].

The fact that the integral equation (1.24) has a solution is not sufficient to insure the existence of the potential Eq. (1.23). It may not be out of place to indicate here how conditions sufficient for the construction of the potential from the spectral function can be obtained.

The only difficulty in justifying the construction occurs in Eq. (1.22) involving second derivatives of $K(r, t)$. It can be dealt with by using a limiting process. We introduce an auxiliary function $\varrho^M(E)$, $M > 0$, defined by

$$\left.\begin{aligned} \varrho^M(E) &= \varrho(E), \qquad E \leqslant M, \\ \frac{d\varrho^M(E)}{dE} &= \frac{d\varrho_1(E)}{dE}, \quad E > M. \end{aligned}\right\} \tag{1.30}$$

Assuming e. g. a comparison potential $V_1(r)$ continuous for $r > 0$ we have no difficulties with the second derivatives of the function

$$g^M(s, t) = \int_{-\infty}^{\infty} d\left[\varrho^M(E) - \varrho_1(E)\right] \varphi_1(E, s)\, \varphi_1(E, t) \tag{1.31}$$

which satisfies the differential equation

$$\frac{\partial^2 g^M}{\partial s^2} - V_1(s)\, g^M = \frac{\partial g^M}{\partial t^2} - V_1(t)\, g^M. \tag{1.32}$$

The function $K^M(r, t)$ constructed by (1.24) using the $g^M(s, t)$ is twice differentiable.

It is easy to verify that the expression

$$\left.\begin{aligned}\chi^M(r,t) \equiv \frac{\partial K^M(r,t)}{\partial r^2} - 2\left(\frac{dK^M(r,r)}{dr}\right) K^M(r,t) \\ -V_1(r)\,K^M(r,t) - \frac{\partial^2 K^M(r,t)}{\partial t^2} + V_1(t)\,K^M(r,t)\end{aligned}\right\} \tag{1.33}$$

satisfies the homogeneous integral equation

$$\chi^M(r,s) + \int_0^r \chi^M(r,t)\,g^M(s,t)\,dt = 0 \tag{1.34}$$

and therefore vanishes identically, so that we have established the equations

$$\left.\begin{aligned}\frac{\partial^2 K^M(r,t)}{\partial r^2} - 2\left(\frac{dK^M(r,r)}{dr}\right) K^M(r,t) - V_1(r)\,K^M(r,t) \\ = \frac{\partial^2 K^M(r,t)}{\partial t^2} - V_1(t)\,K^M(r,t), \\ K^M(r,0) = 0.\end{aligned}\right\} \tag{1.35}$$

It is a consequence of (1.35) that the function

$$\varphi^M(E,r) = \varphi_1(E,r) + \int_0^r K^M(r,t)\,\varphi_1(E,t)\,dt \tag{1.36}$$

satisfies a Schroedinger equation with the potential

$$V^M(r) = V_1(r) + 2\frac{dK^M(r,r)}{dr} \tag{1.37}$$

or, what amounts to the same, the integral equation

$$\varphi^M(E,r) = \varphi_1(E,r) + 2\int_0^r G(E,r,t)\frac{dK^M(t,t)}{dt}\varphi^M(E,t)\,dt, \tag{1.38}$$

where $G(E,r,t)$ is the appropriate Green's function.

We now assume that $\partial g^M(s,t)/\partial t$ is uniformly bounded for all M and in every closed region of the (s,t) plane and that, furthermore,

$$\lim_{M\to\infty} \frac{\partial g^M(s,t)}{\partial t} = \Omega(s,t) \tag{1.39}$$

exists. Then $\Omega(s,t)$ is integrable and

$$\int_0^t \Omega(s,\eta)\,d\eta = \lim_{M\to\infty}\int_0^t \frac{\partial g^M(s,\eta)}{\partial \eta}\,d\eta = \lim_{M\to\infty} g^M(s,t) = g(s,t) \tag{1.40}$$

exists also, and we have

$$\Omega(s,t) = \frac{\partial g(s,t)}{\partial t}. \tag{1.41}$$

The existence of

$$K(r,t) = \lim_{M\to\infty} K^M(r,t) \tag{1.42}$$

which is now a solution of (1.24) is a consequence of simple theorems on integral equations. From the explicit solution of (1.24) follows the existence of

$$\frac{dK(r,r)}{dr} = \lim_{M\to\infty} \frac{dK^M(r,r)}{dr} \tag{1.43}$$

and its integrability. Obviously

$$\lim_{M\to\infty} \varphi^M(E,r) = \varphi(E,r) = \varphi_1(E,r) + \int_0^r K(r,t)\,\varphi_1(E\,t)\,dt \tag{1.44}$$

exists and satisfies

$$\varphi(E,r) = \varphi_1(E,r) + 2\int_0^r G(E,r,t)\,\frac{dK(t,t)}{dt}\,\varphi_1(E,t)\,dt. \tag{1.45}$$

This concludes the verification.

The following conditions on the difference $\varrho(E) - \varrho_1(E)$ and on $V_1(r)$ are sufficient to insure the above conditions and hence the applicability of the construction procedure. They cover most of the physically interesting cases:

$$\int_0^r r^\alpha\,|V_1(r)|\,dr + \int_1^\infty r\,|V_1(r)|\,dr < \infty\ ,\quad 0<\alpha<1 \tag{1.46}$$

$$\left.\begin{aligned} \frac{d}{dE}[\varrho(E) - \varrho_1(E)] &= \frac{\sqrt{E}}{\pi}\left[\frac{\Gamma}{E} + \frac{F(E)}{E^{1+\varepsilon}}\right], \\ E > E_0 > 0,&\quad \varepsilon > 0, \end{aligned}\right\} \tag{1.47}$$

where

$$|F(E)| < \gamma < \infty \tag{1.48}$$

and Γ is constant.

The extension of the foregoing considerations to higher angular momenta, $l>0$, is straightforward. We define the solution $f(k,r)$ as before by

$$\lim_{r\to\infty} e^{ikr} f(k,r) = 1 \tag{1.49}$$

and call

$$f(k) = (2l+1)\lim_{r\to 0} r^l f(k,r). \tag{1.50}$$

Further we define the solution $\varphi(E,r)$ by

$$\lim_{r\to 0} \varphi(E,r)/r^{l+1} = 1. \tag{1.51}$$

Then, for real k,

$$\varphi(k^2, r) = \frac{1}{2ik}[f(k)f(-k, r) - f(-k)f(k, r)] \qquad (1.52)$$

while, in the bound states, $k = -i\varkappa_m$,

$$f(-\varkappa_m, r) = \lim_{l \to 0}\left[\frac{f(-i\varkappa_m, l)}{l^{l+1}}\right]\varphi(-\varkappa_m^2, r). \qquad (1.53)$$

Appendix II of ref. 1 can now be worked through for $l > 0$ (dropping, however, the distinction between φ and $\bar{\varphi}$) and gives for $d\varrho/dE$ (cf. Eq. (1.11) and ref. 1, Eq. (A 2.37))

$$\left.\begin{aligned} \frac{d\varrho}{dE} &= \sum C_m \delta(E - E_m), \quad E < 0 \\ \frac{d\varrho}{dE} &= \frac{\sqrt{E}}{\pi |f(\sqrt{E})|^2} \quad , \quad E \geqslant 0 \end{aligned}\right\} \qquad (1.54)$$

where

$$C_m = \int_0^\infty [\varphi(-\varkappa_m^2, r)]^2 \, dr = \frac{2i\varkappa_m}{\dot{f}(-i\varkappa_m)} \lim_{r \to 0}[f(-i\varkappa_m, r)/r^{l+1}]. \qquad (1.55)$$

With these new definitions one arrives again at the basic equations (1.23), (1.24) and (1.25). [6].

§ 2. Two Applications of the Integral Equation.

(a) *Equivalent Potentials.*

We first want to exhibit the dependence of the manifold of equivalent potentials (i. e. potentials with the same S-phase shift and bound states) on the parameters C_l. We choose for $V_1(r)$ an arbitrary potential out of this manifold corresponding to the values C_{1l}. Since equivalent potentials have the same $f(k)$, we get

$$\frac{d[\varrho(E) - \varrho_1(E)]}{dE} = 0, \quad E > 0 \qquad (2.1)$$

and

$$\frac{d[\varrho(E) - \varrho_1(E)]}{dE} = \sum_{l=1}^{m} [C_l - C_{1l}]\,\delta(E - E_l), \quad E < 0. \qquad (2.2)$$

The kernel (1.25) then reduces to a finite bilinear series

$$g(s, t) = \sum_{l=1}^{m} (C_l - C_{1l})\ \varphi_{1l}(s)\, \varphi_{1l}(t), \tag{2.3}$$

where we have used the abbreviation

$$\varphi_{1l}(r) \equiv \varphi_1(E_l, r) \tag{2.4}$$

for the bound state wave functions. To solve the integral equation (1.24) we make the Ansatz

$$K(r, s) = \sum_{l=1}^{m} a_l(r)\, \varphi_{1l}(s). \tag{2.5}$$

The $a_l(r)$ are then determined by the linear equations

$$\sum_{l=1}^{m} M_{kl}(r)\, a_l(r) = (C_k - C_{1k})\, \varphi_{1k}(r), \tag{2.6}$$

where

$$M_{kl}(r) = -\delta_{kl} - (C_k - C_{1k}) \int_0^r \varphi_{1k}(t)\, \varphi_{1l}(t)\, dt. \tag{2.7}$$

As we have seen at the end of § 1, the matrix M_{kl} has an inverse, so that

$$a_j(r) = \sum_{k=1}^{m} M_{jk}^{-1}(r)\, (C_k - C_{1k})\, \varphi_{1k}(r) \tag{2.8}$$

and

$$\left.\begin{aligned} K(r, s) &= \sum_{j,k} \varphi_{1j}(s)\, M_{jk}^{-1}(r)\, (C_k - C_{1k})\, \varphi_{1k}(r) \\ V(r) &= V_1(r) - 2\, \frac{d^2}{dr^2} \log \operatorname{Det} \| M_{jk}(r) \|. \end{aligned}\right| \tag{2.9}$$

Examples of equivalent potentials have been given in ref. 3.

(b) *Independence of Binding Energies from the Phase Shift.*

The independence of the binding energies from the phase shift is a simple consequence of the Gel'fand-Levitan theory. We consider the S-phase shift as given and assume that there exists a potential $V_1(r)$ reproducing this phase shift and having a bound state at the position $k = -i\varkappa_1$. We now make the assumption that $V_1(r)$ satisfies Eq. (1.46) from which it follows easily that

$$\frac{1}{|f_1(k)|^2} = 1 + \frac{F(k)}{k^{1-\alpha}}, \quad |F(k)| < N < \infty . \tag{2.10}$$

(2.10) is satisfied in all physically interesting cases.

Next we construct the $f(k)$ corresponding to the same phase, but the bound state at a different position, $k = -i\varkappa$:

$$f(k) = f_1(k)\frac{k^2+\varkappa^2}{k^2+\varkappa_1^2} \tag{2.11}$$

so that

$$\left.\begin{aligned} \frac{d}{dE}[\varrho(E)-\varrho_1(E)] &= C\delta(E+\varkappa^2) - C_1\delta(E+\varkappa_1^2), \\ & \qquad\qquad E<0; \\ \frac{d}{dE}[\varrho(E)-\varrho_1(E)] &= \frac{\sqrt{E}}{\pi}\frac{1}{|f_1(\sqrt{E})|^2}\frac{(2E+\varkappa_1^2+\varkappa^2)(\varkappa_1^2-\varkappa^2)}{(E+\varkappa^2)^2}, \\ & \qquad\qquad E>0. \end{aligned}\right\} \tag{2.12}$$

Evidently the conditions (1.46) and (1.48) are satisfied so that a potential corresponding to the same phase shift, arbitrary position of the bound state and arbitrary constant C can be constructed.

For the case when the position of a bound state is changed from $-i\varkappa_1$ to $-i\varkappa$ the $g(s,t)$ can be explicitly calculated.

By (1.25) and (2.12) we have

$$g(s,t) = C\varphi_1(-\varkappa^2,s)\varphi_1(-\varkappa^2,t) - C_1\varphi_1(-\varkappa_1^2,s)\varphi_1(-\varkappa_1^2,t) + g^{(c)}(s,t), \tag{2.13}$$

where the contribution $g^{(c)}(s,t)$ from the continuum is

$$g^{(c)}(s,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{k^2dk}{|f_1(k)|^2}\left(\frac{2(\varkappa_1^2-\varkappa^2)}{k^2+\varkappa^2} + \frac{(\varkappa_1^2-\varkappa^2)^2}{(k^2+\varkappa^2)^2}\right)\varphi_1(k^2,s)\varphi_1(k^2,t). \tag{2.14}$$

By (1.8) and use of contour integration this becomes, for $s \geqslant t$,

$$\left.\begin{aligned} g^{(c)}(s,t) &= \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{k\,dk}{f_1(k)}\left(\frac{2(\varkappa_1^2-\varkappa^2)}{k^2+\varkappa^2} + \frac{(\varkappa_1^2-\varkappa^2)^2}{(k^2+\varkappa^2)^2}\right) f_1(k,s)\varphi_1(k^2,t) \\ &= \sum_{l=1}^{m} C_l \frac{(\varkappa_1^2-\varkappa^2)(2\varkappa_l^2-\varkappa^2-\varkappa_1^2)}{(\varkappa_l^2-\varkappa^2)^2}\varphi_1(-\varkappa_l^2,s)\varphi_1(-\varkappa_l^2,t) \\ &\quad + \frac{i}{2\varkappa}\frac{\partial}{\partial k}\left[\frac{(k^2+\varkappa_1^2)^2}{f_1(k)} f_1(k,s)\varphi_1(k^2,t)\right]_{k=-i\varkappa}. \end{aligned}\right\} \tag{2.15}$$

For $s < t$, $g(s, t)$ can be determined from the symmetry relation $g(s, t) = g(t, s)$.

It is interesting to calculate the derivative of the potential with respect to the variation of one of the binding energies. From (1.23) and (1.24) we have

$$\left.\begin{aligned}\left[\frac{\partial}{\partial \varkappa} V(r;\varkappa)\right]_{\varkappa=\varkappa_1} &= 2\frac{\partial}{\partial r}\left[\frac{\partial}{\partial \varkappa} K(r, r;\varkappa)\right]_{\varkappa=\varkappa_1} \\ &= -2\frac{\partial}{\partial r}\left[\frac{\partial}{\partial \varkappa} g(r, r;\varkappa)\right]_{\varkappa=\varkappa_1}\end{aligned}\right\} \tag{2.16}$$

and by (2.15)

$$\left.\begin{aligned}\left[\frac{\partial}{\partial \varkappa} g(r, r;\varkappa)\right]_{\varkappa=\varkappa_1} &= -\frac{1}{4}\sum_{l=1}^{m} A_l [f_1(-i\varkappa_l, r)]^2 \\ &\quad - \frac{2 i\varkappa_1}{\dot{f}_1(-i\varkappa_1) f_1'(-i\varkappa_1, 0)}\left[\frac{\partial}{\partial \varkappa}[f(-i\varkappa, r)]^2\right]_{\varkappa=\varkappa_1}.\end{aligned}\right\} \tag{2.17}$$

Thus $[\partial V(r;\varkappa)/\partial\varkappa]_{\varkappa=\varkappa_1}$, has the form

$$\left.\begin{aligned}\left[\frac{\partial V(r;\varkappa)}{\partial \varkappa}\right]_{\varkappa=\varkappa_1} &= \sum_{l=1}^{m} A_l \dot{f}_1(-i\varkappa_l, r) f_1'(-i\varkappa_l, r) \\ &+ \frac{8 i\varkappa_1}{\dot{f}_1(-i\varkappa_1) f_1'(-i\varkappa_1, 0)}\left[\frac{\partial}{\partial \varkappa}(f_1(-i\varkappa, r) f_1'(-i\varkappa, r)\right]_{\varkappa=\varkappa_1}.\end{aligned}\right\} \tag{2.18}$$

This expression can be checked in an elementary way. We consider the infinitesimal change of potential

$$\delta V(r) = \delta\varkappa\left[\frac{\partial V(r;\varkappa)}{\partial \varkappa}\right]_{\varkappa=\varkappa_1} \tag{2.19}$$

and shall verify that to first order in $\delta\varkappa$ it changes only the binding energy $-\varkappa_1^2$, to $-(\varkappa_1 + \delta\varkappa)^2$, leaving the other binding energies and phase shift unaltered.

We have seen in ref. 3, Eq. (2,3), that the first term of (2.18) does not change the eigenvalues or phase shift; it merely leads one to a neighbouring "equivalent" potential. Next we calculate the change of the eigenvalues due to the second term in (2.18). Let ψ_l be the normalized eigenfunction of V_1, belonging to binding energy E_l. Then

$$\left.\begin{aligned}\delta E_l &= \int_0^\infty \delta V(r)\,[\psi_l(r)]^2\,dr \\ &= \delta\varkappa\,\frac{8 i\varkappa_1}{\dot{f}_1(-i\varkappa_1) f_1'(-i\varkappa_1, 0)}\int_0^\infty\left[\frac{\partial}{\partial \varkappa}(f_1(-i\varkappa, r) f_1'(-i\varkappa, r)\right]_{\varkappa=\varkappa_1}\psi_l^2(r)\,dr.\end{aligned}\right\} \tag{2.20}$$

With the help of the Schroedinger equation we find (cf. ref. 3, Eq. (2.3))

$$\int_0^\infty f_1(-i\varkappa, r)\, f_1'(-i\varkappa, r)\, \psi_e^2(r)\, dr = \frac{1}{4}(\varkappa^2 + E_m)\left[\int_0^\infty f_1(-i\varkappa, r)\, \psi_l(r)\, dr\right]^2. \quad (2.21)$$

Differentiating with respect to $\varkappa$ and substituting into (2.20) gives, with the help of (1.10) and (1.15),

$$\left.\begin{aligned} \delta E_e &= \delta\varkappa \frac{8 i\varkappa_1}{\dot{f}_1(-i\varkappa_1)\, f_1'(-i\varkappa_1, 0)} \frac{\varkappa_1}{2}\left[\int_0^\infty \frac{f_1'(-i\varkappa, 0)}{C_1^{1/2}} \psi_1(r)\, \psi_l(r)\, dr\right] \\ &= \delta_{l1}\left[(\varkappa_1 + \delta\varkappa)^2 - \varkappa_1^2\right]. \end{aligned}\right\} \quad (2.22)$$

Similarly one can check that $\delta V(r)$ does not alter the phase shift, which completes the verification.

As an illustration, we shall explicitly construct the potentials corresponding to the phase shift $\eta(k)$ given by

$$\left.\begin{aligned} k \cot \eta(k) &= -\alpha + \frac{1}{2} r_0 k^2 \\ \alpha > 0, \quad r_0 > 0, \quad 2\alpha r_0 &< 1, \end{aligned}\right\} \quad (2.23)$$

and a bound state located at the arbitrary negative energy $E_1 = \varkappa_1^2$. Eq. (2.23) has been used for an approximate description of low energy neutron proton scattering in the triplet state. According to Eq. (6.19) of ref. 1 this leads to

$$f(k) = \frac{4k^2 + 4\varkappa_1^2}{[2k - i(\varrho - \sigma)]\,[2k - i(\varrho + \sigma)]}, \quad (2.24)$$

where

$$\varrho = \frac{2}{r_0}, \qquad \sigma = \frac{2}{r_0}\sqrt{1 - 2\alpha r_0}. \quad (2.25)$$

As auxiliary potential $V_1(r)$ we chose the potential corresponding to

$$f_1(k) = \frac{(2k - 2i\varkappa_1)^2}{[2k - i(\varrho - \sigma)]\,[2k - i(\varrho + \sigma)]}. \quad (2.26)$$

This potential has no bound state and is uniquely defined. It is contained among the examples of BARGMANN [7]. Since

$$|f_1(k)|^2 = |f(k)|^2 \quad (2.27)$$

the kernel $g(s, t)$ of Eq. (1.25) reduces to a single bilinear term so that the solution of the integral equation (1.24) becomes trivial and $V(r)$ can be determined.

To obtain an expression for $V_1(r)$ we express $\varkappa_1$ by means of a new parameter λ as

$$\varkappa_1 = \frac{1}{2}(\varrho+\sigma)\frac{1+\lambda}{1-\lambda}, \quad -1<\lambda<1. \text{ [8]}. \tag{2.28}$$

Then $V_1(r)$ can be written as

$$\left.\begin{aligned} V_1(r) = {} & 2\varrho\sigma\{(\varrho+\lambda\sigma)^2(\sigma+\lambda\varrho)^2(\varrho-\sigma)^2[e^{\frac{1}{2}(\varrho+\sigma)r}-\lambda^2 e^{-\frac{1}{2}(\varrho+\sigma)r}]^2 \\ & -\lambda^2(\varrho+\sigma)^2[(\varrho+\lambda\sigma)^2 e^{\frac{1}{2}(\varrho-\sigma)r}-(\sigma+\lambda\varrho)^2 e^{-\frac{1}{2}(\varrho-\sigma)r}]^2\} \\ & \times\{(\varrho+\lambda\sigma)^2[(\sigma e^{\varrho r}+\varrho\lambda^2 e^{-\sigma r}]-(\sigma+\lambda\varrho)^2[\varrho e^{\sigma r}+\sigma\lambda^2 e^{-\varrho r}]\}^{-2} \end{aligned}\right\} \tag{2.29}$$

and the corresponding $\varphi_1(E_1, r)$ is

$$\left.\begin{aligned} \varphi_1(E_1, r) = {} & e^{\varkappa_1 r}(1-\lambda^2) \\ & \times\frac{(\varrho+\lambda\sigma)[\sigma e^{\varrho r}+\varrho\lambda e^{-\sigma r}]-(\sigma+\lambda\varrho)[\varrho e^{\sigma r}+\lambda\sigma e^{-\varrho r}]}{(\varrho+\lambda\sigma)^2[\sigma e^{\varrho r}+\varrho\lambda^2 e^{-\sigma r}]-(\sigma+\lambda\varrho)^2[\varrho e^{\sigma r}+\lambda^2\sigma e^{-\varrho r}]}. \end{aligned}\right\} \tag{2.30}$$

The integral equation for $K(r, t)$ is

$$\left.\begin{aligned} & K(r, s)+C\,\varphi_1(E_1, r)\,\varphi_1(E_1, s) \\ & +C\,\varphi_1(E_1, s)\int_0^r K(r, t)\,\varphi_1(E_1, t)\,dt = 0 \end{aligned}\right\} \tag{2.31}$$

with the solution

$$K(r, t) = \frac{C\varphi_1(E_1, r)\,\varphi_1(E_1, t)}{1+C\int_0^r[\varphi_1(E_1, t')]^2\,dt'}, \quad 0<C<\infty\,. \tag{2.32}$$

The family of potentials with the phase shift (2.10) and the bound state at the arbitrary energy $E_1<0$ is therefore given by

$$V(r) = V_1(r)-2C\frac{\partial}{\partial r}\frac{[\varphi_1(E_1, r)]^2}{1+C\int_0^r[\varphi_1(E_1, t)]^2\,dt}. \tag{2.33}$$

Summarizing we may note that we have here a four parameter manifold of potentials; ϱ and σ are determined by the phase shift, λ by the position of the bound state and C by the normalization of the bound state function $\varphi(E_1, r)$.

§ 3. Remarks on the Interpretation of Scattering Data.

In some of the early investigations of the S-matrix, it was believed that the zeros of the S-matrix determine the bound states. Counter-examples showed, however, that in some cases, at the so-called false zeros, no bound states occur. Still, the idea survived that at least the S-matrix must always vanish at the points of the discrete spectrum. For instance, this idea was used in one of the justifications of the so-called effective range theory [9], to establish the connection between the neutron proton triplet S-phase and the binding energy of the deuteron. We have seen that such a connection does not exist, in general. We shall show that only for sufficiently short-range potentials do bound states necessarily coincide with the zeros of the S-matrix. (It should be noted that the original derivation of the effective range formalism by SCHWINGER [10] does not make use of this connection, but is based only on the assumption that the range of the potential is short compared to the size of the deuteron).

Thus consider the case where the S-matrix, $S(k) = e^{2i\eta(k)}$, is known and has zeros at $k = -i\varkappa_l$, $l = 1, 2, \cdots m$. Suppose further we know *a priori* that the underlying potential is short range in the sense that

$$\lim_{r\to\infty} e^{2\varkappa r} V(r) = 0 \tag{3.1}$$

for some $\varkappa > 0$. Then $f(k)$ is regular in $Im(k) < \varkappa$ so that in the strip, $-\varkappa < Im(k) < 0$, there is a one to one correspondence between zeros of the S-matrix and binding energies. On the other hand, no bound states need occur at those zeros $-i\varkappa_l$ which lie below this strip. For suppose there exists a potential V_1 reproducing the phase shift and having bound states at $-i\varkappa_l$. According to (2.19) and (2.18) one can add an infinitesimal increment $\delta V = \varepsilon\,[(\partial/\partial\varkappa)\,(f_1(-i\varkappa, r)\,f_1'(-i\varkappa, r))]_{\varkappa=\varkappa_l}$, which changes only the position of the l'th bound state. Since it behaves asymptotically like $\delta V \sim \varepsilon\,(2\varkappa_l r - 1)\exp(-2\varkappa_l r)$ it does not violate (3.1), provided that $\varkappa_l > \varkappa$, and hence leads to an acceptable potential with the same S-matrix, but a displaced bound state.

For the neutron proton system $\varkappa$, Eq. (3.1) may be estimated by meson theory and is substantially larger than the smallest

zero, $-i\varkappa_1$, of the S-matrix ($\varkappa/\varkappa_1 \sim 1{\cdot}5$). From this additional information it follows that this zero corresponds to the deuteron binding energy.

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References & Footnotes.

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[3] R. Jost and W. Kohn, Phys. Rev. **88**, 382 (1952); See also B. Holmberg, Nuovo Cimento **9**, 597 (1952).
[4] The derivation of (1.23) by means of (1.22) is not mathematically strict, because of the occurrence of second derivatives in (1.22). Conditions for the validity of (1.23) will be given on page 8 ff.
[5] One can verify the completeness of the functions $\varphi(E, r)$ defined by (1.12) with the aid of (1.11), (1.12), and (1.13).
[6] The connection of $f(k)$ and $\eta(k)$, in the absence of bound states, is given by

$$f(k) = \frac{1 \cdot 1 \cdot 3 \cdots (2l+1)}{(ik)^l} \exp\left[-\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\eta(k')}{k'-k'} dk'\right],$$

with obvious modifications when bound states are present ([1], Eqs. (2.15)—(2.18)).
[7] V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949), Eqs. (4.1)—(4.6).
[8] The connection between our parameter λ and Bargmann's α and β (ref. 5, Eq. (4.3)) is $\beta = \lambda^2$, $\alpha = \left(\frac{\sigma + \lambda\varrho}{\varrho + \lambda\sigma}\right)^2$.
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Note added in proof.

After completion of this manuscript the existence of a more detailed paper by I. M. Gel'fand and B. M. Levitan (Isvestiia Akad. Nauk. S.S.S.R. **15**, 309, 1951) was brought to our attention. This paper contains all the proofs and treats, also the case of boundary conditions $\varphi(0) = 0$. No application to scattering theory is discussed.

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